Analysis of Waveguide Discontinuities Using Edge Elements in a Hybrid Mode Matching/Finite Elements Approach

D. Arena, M. Ludovico, G. Manara, *Senior Member, IEEE*, and A. Monorchio, *Member, IEEE*

***Abstract*—A mode matching (MM)/finite element method (FEM) for the analysis of waveguide discontinuities is presented. The hybrid approach described combines the computational efficiency of the modal analysis with the versatility and flexibility of FEM and enables us to accurately model arbitrary cross section waveguides, where modal expansions cannot be derived analytically. The proposed procedure is based on the edge element expansion of the transverse field components for the direct calculation of the coupling integrals involved in the MM formulation. Numerical and experimental results are presented to show the validity and the accuracy of the method.**

***Index Terms*—Finite elements method, hybrid methods, mode matching.**

I. Introduction

Due to its numerical efficiency, mode matching (MM) analysis has been widely employed for designing waveguide components, such as transformers, polarizers, couplers, filters, and multiplexers. This method is well-suited to handle geometries where modal expansions of electromagnetic fields can be derived analytically. Conversely, the finite element method (FEM) is well-suited for the analysis of complex and irregular structures, since it employs an unstructured mesh that conforms to these geometries.

The objective of this paper is to present a hybrid scheme that combines the above two methods in order to retain the advantages of both techniques. In particular, MM is used all over that part of the computational domain where an analytical solution is known, while waveguides with arbitrary cross section are analized by means of a FEM approach. Examples of hybridization of the two methods have been recently proposed in the literature. In [1] and [2], a hybrid MM/FEM technique is proposed for homogeneous waveguide problems, based on a nodal functions expansion of scalar potentials [3]. In [4] and [5], an approach combining MM with both two-dimensional (2-D) and three-dimensional (3-D) formulations of FEM is introduced, using covariant projection quadrilateral elements [6]. The aim of this work is to exploit the effectiveness and the accuracy of a different approach based on a standard Delaunay triangulation of the analysis domain and on the use of edge elements for the expansion of the transverse field components. Edge elements have proven effective for the calculation of waveguide eigenfunctions even in the presence of dishomogeneities [7] and field singularities [8]. In our case, Whitney's elements are used in formulating the FEM, providing a straightforward procedure for the calculation of the coupling integrals involved in waveguide discontinuities analysis.

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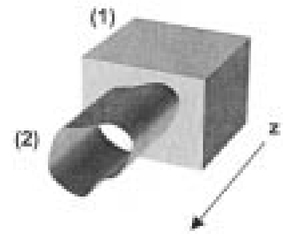

Fig. 1. Waveguide stepped discontinuity problem.

II. Formulation

The geometry of the problem is depicted in Fig. 1. It represents a boundary reduction discontinuity between a standard (rectangular or circular) waveguide (region 1) and an arbitrary cross section homogeneous waveguide (region 2). The approach, however, can be easily generalized to every kind of arbitrary cross section to arbitrary cross section waveguide step. The aim is that of computing the generalized scattering matrix (GSM) of the waveguide discontinuity. It can be obtained by calculating the following integrals [9]

$$A_{m,n} = \int\int_{S_2} \left(\underline{e}_n^{(1)} \times \underline{h}_m^{(2)^*}\right) \cdot \underline{i}_z dS \tag{1a}$$

$$P_n^{(i)} = \int\int_{S_i} \left(\left(\underline{e}_n^{(i)} \times \underline{h}_n^{(i)^*}\right) \cdot \underline{i}_z dS \tag{1b}$$

where

- i_z unit vector in the propagation direction;
- $\underline{e}_n^{(i)}$ electric vector eigenfunction of the nth mode in the ith guide;
- $\underline{h}_n^{(i)}$ magnetic vector eigenfunction of the nth mode in the ith guide;
- S_i cross section of the ith guide.

We note that, while the field components in waveguide 1 can be derived analytically, the modal expansions of fields in wave-

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guide 2 are calculated through a FEM solution of the Helmholtz vector wave equation

$$\nabla_t \times (\nabla_t \times \underline{F}) - \left(k_0^2 - \beta^2\right) \underline{F} = 0. \tag{2}$$

In (2), ∇_t denotes the two-dimensional gradient operating on the coordinates of the transverse plane, k_0 is the free-space propagation constant, β is the phase constant along the z-direction, and $\underline{F}$ coincides with either the transverse electric field $\underline{E}_t$ or the transverse magnetic field $\underline{H}_t$ for TE and TM modes, respectively. Air-filled ($\varepsilon_\mathrm{r} = 1$ and $\mu_\mathrm{r} = 1$) waveguides have been assumed for the sake of simplicity.

The field is, therefore, expanded over the waveguide cross section using the edge-based Whitney's vector functions $\underline{W}(x,y)$. By introducing a set of test functions $\underline{T}(x,y)$, after some manipulations, we obtain the so-called weak form of (2) that can be discretized by using a triangular grid and the edge-based field expansion functions. Additionally, a Galerkin procedure, i.e., $\underline{T}(x,y) = \underline{W}(x,y)$, is employed. Assembly of the element equations yields the global matrix equation system [10]

$$\underline{\underline{K}}_\nabla \cdot \underline{F} = \gamma^2 \underline{\underline{K}} \cdot \underline{F} \tag{3}$$

where the entries of the matrices can be cast in the following form

$$k_{\nabla ij} = \int\limits_{S_2}\int (\nabla_t \times \underline{W}_i) \cdot \left(\nabla_t \times \underline{W}_j\right) dxdy$$
$$k_{ij} = \int\limits_{S_2}\int \underline{W}_i \cdot \underline{W}_j dxdy \tag{4}$$

with ij representing the global numbering of edges. It is apparent that (3) represents a standard generalized eigenvalue problem. In particular, eigenvalues γ_i^2 correspond to the square of the modal transverse eigenvalues (cut-off numbers), while eigenvectors $\underline{F}_i$ denote the values of the transverse field along the edges of the triangular mesh corresponding to the i-th mode. For TM modes ($\underline{F} = H_t$), the number of unknowns equals that of global edges, whereas for TE modes ($\underline{F} = E_t$) the number of unknowns equals the number of inner edges, being $\underline{E}_t$ identically zero at boundary edges.

To keep advantage of the sparse nature of the generalized eigenvalue problem (3), an implicitly restarted Arnoldi algorithm [11] can be suitably employed in order to evaluate the lowest nonvanishing eigenvalues and the corresponding eigenfunctions. This approach provides an approximated representation of modal fields $\underline{e}_n^{(2)}$ and $\underline{h}_n^{(2)}$ to be used in coupling integrals calculation. Since transverse electric and magnetic fields are directly available for TE and TM modes respectively, (1) can be rewritten as

$$A_{m,n}^{(\mathrm{TE/TE})} = \frac{K_z^{(2)^*}}{\omega\mu} \int\limits_{S_2}\int \left(\underline{e}_n^{(1)} \cdot \underline{e}_m^{(2)^*}\right) dS \tag{5a}$$

$$A_{m,n}^{(\mathrm{TM/TM})} = \frac{K_z^{(1)}}{\omega\varepsilon} \int\limits_{S_2}\int \left(\underline{h}_n^{(1)} \cdot \underline{h}_m^{(2)^*}\right) dS \tag{5b}$$

$$A_{m,n}^{(\mathrm{TE/TM})} = \int\limits_{S_2}\int \underline{e}_n^{(1)} \cdot \left(\underline{h}_m^{(2)^*} \times i_z\right) dS. \tag{5c}$$

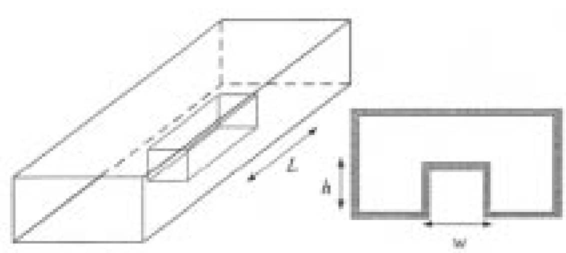

Fig. 2. Geometry of the junction under investigation ($L = 20$ mm, $w = 6$ mm, $h = 4.5$ mm).

The terms $\mathrm{TM/TE}$ can be proved to be identically zero [12]. An analogue procedure can be used for the integrals $P_n^{(i)}$ defined in (1b). Finally, we point out that, for evaluating the frequency independent coupling integrals in (5), a standard Gauss bidimensional quadrature formula, employing three points per element can be satisfactorily used. A frequency sweep can be obtained simply updating the multiplying factors depending on K_z.

Once the GSM of the single discontinuities have been evaluated, the analysis of an overall component can be carried out as a sequence of scattering matrix connections and reference plane shifts.

It is worth observing that the accuracy of the MM/FEM analysis is mainly affected by two different factors, i) the number of triangles constituting the mesh grid and their geometrical properties and ii) the number of modes used for representing the fields at waveguides discontinuities.

In order to avoid the well-known problem of relative convergence [13], we selected the appropriate TE and TM expansions by means of the spectral criterion, briefly summarized by the following points:

1) for each expansion (TE and TM) and for each side of the discontinuity, modes are searched and ordered by increasing cut-off frequency;
2) the number of modes is chosen in order to have approximately the same cut-off wave number $k_c^{\max}$ for the highest mode considered in each TE or TM expansion.

For nearly-uniform meshes, the following reference criterion for determining the value of $k_c^{\max}$ has been adopted

$$k_c^{\max} < \alpha \frac{\pi}{l_\mathrm{av}} \tag{6}$$

where l_av is the average length of the edge constituting the mesh. A value ranging from 0.5 to 0.7 must be assumed for α. This expression relies on the consideration that the element size has to be sufficiently small with respect to the smallest cut-off wavelength of the relevant modes in order to interpolate correctly tangential fields.

III. Numerical Results

Some numerical results are presented in this section to demonstrate the effectiveness of the hybrid technique proposed.

As a first step, we analyzed the junction in Fig. 2, where a standard WR75 rectangular waveguide is connected on both sides to a ridged waveguide 20 mm long. The metallic ridge is 6 mm wide and has a height of 4.5 mm. The ridged waveguide

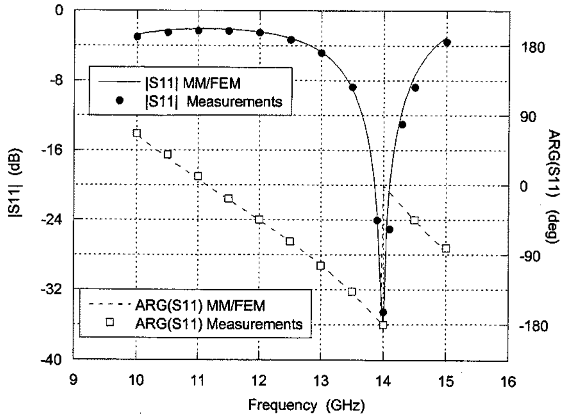

Fig. 3. Amplitude and phase of S_{11} versus frequency for the junction in Fig. 2.

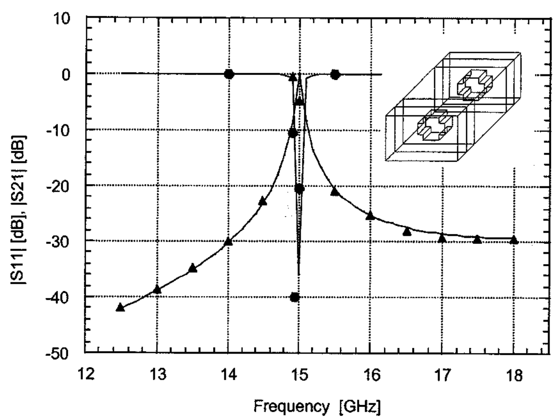

Fig. 4. Transmission and reflection coefficients for the cross-iris coupled filter (continuous line: calculated S_{21}; dashed line: calculated S_{11}; dots and triangles: [13]).

has been meshed by using a triangular grid comprising 344 elements with 550 edges (70 boundary edges). For the grid considered, the expression (6) gives a boundary of 2 mm^{-1} for the value of $k_c^{\max}$. By selecting the modes according to the spectral criterion and fixing $k_c^{\max} = 1.8$ mm^{-1}, the first 50 TE modes and 31 TM modes have been taken into account into the ridged waveguide, while the first 55 TE modes and 40 TM modes have been considered into the rectangular waveguide. The S_{11} scattering parameter obtained in this case is plotted in Fig. 3. The analysis has taken about 40 s for 20 frequency samples on a 250 MHz RISC processor of a SGI Origin 2000. To have a term of comparison, the same structure has been analyzed with the commercial software Ansoft HFSS, based on a full 3D FEM formulation. A 3012 tetrahedral elements mesh was employed and the required CPU time on the above-mentioned processor was about 4 min for a 20 points fast-frequency sweep.

As a second example, in Fig. 4 we present the calculation of the transmission and reflection coefficients of the WR 62 cross-iris coupled filter described in [14]. In this case, the cross irises have been discretized using a mesh of 154 elements (249 edges). The corresponding $k_c^{\max}$ boundary was 2.2 mm^{-1} leading to 11 TE and 5 TM modes in the cross irises and, consequently, 56 TE and 40 TM modes in the rectangular waveguides. The accordance between measured and computed parameters obtained for both the case-studies confirms the accuracy and the effectiveness of the hybrid technique proposed.

IV. Conclusions

A hybrid technique for solving waveguide discontinuity problems has been presented; it employs an edge-based finite element approach for determining modal expansions in the case of arbitrary shaped waveguides and an MM procedure for the evaluation of the GSM of a discontinuity. Some numerical and experimental results have been reported in order to show the accuracy and the computational efficiency of the proposed method.

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